Class 10, given on Jan 25, 2010, for Math 13, Winter 2010

We've learned quite a bit about double integrals in this class. They allow us to calculate the volume of solids over various regions in the xy plane, and also have important applications to physics and engineering. We will spend the next few classes studying *triple integrals*. Although there are no new concepts which arise when moving from double integrals to triple integrals, the complexity of the calculations does increase.

1. Introduction: Triple Integrals over rectangular prisms

Suppose we have a function of three variables $f(x, y, z)$. We cannot visualize the graph of this function anymore, although the graph of f does live in a four-dimensional space, being the set of (x, y, z, w) satisfying $w = f(x, y, z)$. Nevertheless, there are many situations where functions of three variables occur naturally; for example, $f(x, y, z)$ might represent physical quantities such as temperature, density, pressure, etc.

When studying double integrals, we integrate over a domain D which is a subset of the xy plane. Therefore, it should come as no surprise that when we evaluate a triple integral, we do so over a domain D in xyz space. We quickly sketch how triple integrals are defined, skipping a lot of the details since they are very similar to how these calculations were performed when studying double integrals.

We began our study of double integrals by integrating only over rectangles $R = [a, b] \times [c, d]$. Therefore, we should begin by defining triple integrals over rectangular prisms $R = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$, which is the set of points (x, y, z) satisfying $x_1 \le x \le x_2, y_1 \le y \le y_2, z_1 \le z \le z_2$. If we subdivide this rectangular prism into lots of smaller rectangular prisms, then we can calculate the Riemann sum

$$\sum f(x^*, y^*, z^*)\Delta x \Delta y \Delta z,$$

which is supposed to represent the 'four-volume' of the four-dimensional solid over the rectangular prism R and under the graph $w = f(x, y, z)$. Even though we are unable to visualize four dimensions, the analogy with the one and two dimensional cases should make it clear that these Riemann sums should be approximating some sort of volume.

We take the limit of these Riemann sums as the subdivisions of R become finer and finer (namely, as $\Delta x, \Delta y, \Delta z \to 0$), and call the limit, if it exists, the *definite integral* of $f(x, y, z)$ over R, and write it as

$$\iiint\limits_R f(x, y, z)\, dV.$$

The dV stands for a differential volume element.

In practice, we evaluate triple integrals by evaluating iterated integrals, just like how we evaluate double integrals. For example, if we want to integrate $f(x, y, z)$ over $R = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$, we might evaluate an iterated integral such as

$$\iiint\limits_R f(x, y, z)\, dV = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx.$$

The order of integration is from the inside out, so we first integrate with respect to z, and then with respect to y, and finally with respect to x. Under suitable hypotheses on $f(x, y, z)$, a higher dimensional version of Fubini's Theorem is satisfied, which lets us reorder integration. These hypotheses are analogous to the hypotheses required in the two variable case – $f(x, y, z)$ should be continuous except possibly at a finite number of smooth surfaces. Notice that there are now six possible orders of integration, as opposed to the two possible orders in double integration. For instance, if we want to integrate with respect to z, and then x, and then y, we would evaluate the iterated integral

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dx\,dy$$

Example. Let $R = [-1,1]\times[0,2]\times[0,1]$. Evaluate the triple integral

$$\iiint\limits_R x^2y + xe^z\,dV.$$

The integrand is continuous so we can evaluate an iterated integral using whatever order we wish. In this example we will evaluate using the order z, y, and then x; if you wish you can try any of the other five orders to verify that you get the same result.

With the order specified above, the iterated integral we want to evaluate is

$$\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1} x^2y + xe^z\,dz\,dy\,dx.$$

Integrating with respect to z gives

$$\int_{-1}^{1}\int_{0}^{2}\left(x^2yz + xe^z\Big|_{z=0}^{z=1}\right)dy\,dx = \int_{-1}^{1}\int_{0}^{1} x^2y + x(e-1)\,dy\,dx.$$

Integrating this with respect to y gives

$$\int_{-1}^{1}\left(\frac{x^2y^2}{2} + xy(e-1)\Big|_{y=0}^{y=2}\right)dx = \int_{-1}^{1} 2x^2 + 2x(e-1)\,dx.$$

Finally, evaluating the last integral gives

$$\int_{-1}^{1} 2x^2 + 2x(e-1)\,dx = \frac{2x^3}{3} + x^2(e-1)\Big|_{-1}^{1} = \frac{4}{3}.$$

2. Triple integrals over more general regions

Just like how we defined double integrals over not only rectangles, but over more general two-dimensional regions D, we can define triple integrals over three-dimensional regions D whose boundary, which now will be a surface instead of a curve, is sufficiently 'smooth'. In virtually any situation we encounter, the region D will be defined by smooth functions, so the boundary will also be smooth.

In practice, we evaluate double integrals over general regions D by describing D using type I or type II inequalities (if possible), and then calculating iterated integrals

$$\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx, \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

When dealing with three-dimensional regions, we say a region Ω is type I if it can be written as a subset of $\mathbb{R}^3$ defined by inequalities $(x,y)\in D, g_1(x,y)\le z\le g_2(x,y)$, where D is some subset of the xy plane. In other words, Ω can be thought of as a region in between the graphs of two functions $z = g_1(x,y), z = g_2(x,y)$ over the two-dimensional domain D. Then the triple integral of $f(x,y,z)$ over Ω is given by the expression

$$\iiint\limits_\Omega f(x,y,z)\,dV = \iint\limits_D\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dA.$$

In other words, we integrate $f(x,y,z)$ with respect to z first, and then evaluate the resulting expression at the endpoints of integration $z = g_2(x,y), z = g_1(x,y)$. The result will be some function of x and y only, and we calculate the double integral of this function over D using the methods we already know.

Example. Let Ω be the region above the triangle with vertices $(0,0), (0,1)$, and $(1,1)$ in the xy plane, and below the graph of $z = x^2 + y^2$. Evaluate the triple integral

$$\iiint_{\Omega} y\, dV.$$

The description of Ω given to us in this problem is basically a description of the region using type I inequalities. The domain D is the triangle in the xy plane described in the question, and the inequalities on z are $0 \le z \le x^2 + y^2$. Therefore, the triple integral is equal to

$$\iint_{D} \int_0^{x^2+y^2} y\, dz\, dA = \iint_{D} yz\Big|_{z=0}^{z=x^2+y^2} dA = \iint_{D} y(x^2+y^2)\, dA = \iint_{D} x^2y + y^3\, dA.$$

This is a double integral which we now evaluate in the usual way. We first describe D using type I inequalities (notice the ambiguity in terminology here – we are now referring to type I inequalities which describe a subset of the xy plane, not xyz space): $0 \le x \le 1, 0 \le y \le x$. Therefore, the double integral we want to evaluate is equal to

$$\int_0^1 \int_0^x x^2y + y^3 dy\, dx = \int_0^1 \frac{x^2y^2}{2} + \frac{y^4}{4}\Big|_{y=0}^{y=x} dx = \int_0^1 \frac{x^4}{2} + \frac{x^4}{4}\, dx.$$

This integral is equal to

$$\int_0^1 \frac{3x^4}{4}\, dx = \frac{3x^5}{20}\Big|_0^1 = \frac{3}{20}.$$

Notice that we could have also written the original iterated integral as

$$\int_0^1 \int_0^x \int_0^{x^2+y^2} y\, dz\, dy\, dx.$$

As a matter of fact, this is how we will usually write three-fold iterated integrals, instead of writing a double integral and then a single integral.

Of course, as you might expect, there are analogous definitions for type II and type III regions, and integrals over those regions can be calculated in a similar fashion to those in the example above. If a region is simultaneously type I and type II, for example, it is possible to interchange the order of integration, although in practice this is very hard because you will need to draw the region of integration in three dimensions.

3. Applications of triple integrals

All the applications of single and double integrals still hold true for triple integrals. For example, the volume of Ω is given by

$$V(\Omega) = \iiint_{\Omega} dV.$$

The average value of the function $f(x,y,z)$ over Ω is given by

$$\frac{1}{V(\Omega)}\iiint\limits_{\Omega} f(x,y,z)\,dV.$$

If Ω represents the volume filled in by some three dimensional solid, and $\rho(x,y,z)$ is the density at point (x,y,z), then the mass of the solid is given by

$$m = \iiint\limits_{\Omega} \rho(x,y,z)\,dV.$$

The coordinates of the center of mass, $(\overline{x},\overline{y},\overline{z})$, are given by

$$\overline{x} = \frac{1}{m}\iiint\limits_{\Omega} x\rho(x,y,z)\,dV,\ \overline{y} = \frac{1}{m}\iiint\limits_{\Omega} y\rho(x,y,z)\,dV,\ \overline{z} = \frac{1}{m}\iiint\limits_{\Omega} z\rho(x,y,z)\,dV.$$

The moment of inertia about the x-axis is given by

$$I_x = \iiint\limits_{\Omega} (y^2+z^2)\rho(x,y,z)\,dV,$$

with symmetrical formulas for I_y, I_z.

Example. Consider a rectangular prism with edges of length $2w, 2l, 2h$, centered at the origin. Suppose this prism has uniform density. What is the moment of inertia of the box about the x-axis?

We may assume that $\rho(x,y,z) = 1$; then $m = (2w)(2l)(2h) = 8wlh$. The moment of inertia about the x-axis is given by the equation

$$\iiint\limits_{\Omega} (y^2+z^2)\rho(x,y,z)\,dV = \int_{-w}^{w}\int_{-l}^{l}\int_{-h}^{h} (y^2+z^2)\,dz\,dy\,dx.$$

Evaluating this integral yields

$$\begin{aligned}
\int_{-w}^{w}\int_{-l}^{l}\left(y^2 z + \frac{z^3}{3}\Big|_{z=-h}^{z=h}\right) &= \int_{-w}^{w}\int_{-l}^{l} y^2(2h) + \frac{2h^3}{3}\,dy\,dx \\
= \int_{-w}^{w}\left(\frac{y^3(2h)}{3} + \frac{2h^3}{3}y\right)\Big|_{y=-l}^{y=l} dx \\
= \int_{-w}^{w}\frac{4l^3h}{3} + \frac{4lh^3}{3}\,dx = 2w\left(\frac{4l^3h}{3} + \frac{4lh^3}{3}\right).
\end{aligned}$$

If we use the fact that $m = 8wlh$, and then factor this out of the expression above, we obtain

$$I_x = \frac{m}{3}(l^2+h^2),$$

which agrees with the formulas found in books (taking into account the fact that we have let $2l, 2w, 2h$ be the lengths of the sides, not l, w, h.)

Example. Use a triple integral to calculate the volume of a tetrahedron with vertices at $(2,0,0), (0,1,0), (0,0,3)$, and $(0,0,0)$.

A quick sketch of this tetrahedron shows that it is bounded below by a triangle on the xy plane, with vertices at $(0,0), (2,0), (0,1)$, and above by a plane. We begin by determining a formula for this plane. If we wanted to, we could form two vectors on the plane and then take the cross product, but in this case there is actually a faster way to determine the equation of the plane. We know the equation of the plane has the form $ax+by+cz=d$, for some scalars a,b,c,d. Since $(2,0,0),(0,1,0),(0,0,3)$ lie on this plane, these scalars satisfy the equalities

$$2a=d, b=d, 3z=d.$$

Therefore, we might choose $a=3, b=6, z=2, d=6$, and we find an equation $3x+6y+2z=6$. As a function of z, this is $z=3-3x/2-3y$.

The triangle D which bounds the lower part of the tetrahedron is given by inequalities $0 \leq x \leq 2, 0 \leq y \leq 1-x/2$. Therefore, the volume of the tetrahedron is given by the iterated integral

$$\int_0^2 \int_0^{1-x/2} \int_0^{3-3x/2-3y} dz\, dy\, dx.$$

Evaluating this iterated integral gives

$$\begin{aligned} & \int_0^2 \int_0^{1-x/2} \left(3 - \frac{3x}{2} - 3y\right) dy\, dx \\ = & \int_0^2 \left(3y - \frac{3xy}{2} - \frac{3y^2}{2}\right) \Big|_{y=0}^{y=1-x/2} dx = \int_0^2 \frac{3}{2}\left(1-\frac{x}{2}\right)^2 dx \\ = & -\left(1-\frac{x}{2}\right)^3 \Big|_0^2 = 1. \end{aligned}$$

Notice that this is the answer which geometry would have given us. Furthermore, notice that the double integral we get after evaluating the integral with respect to z is exactly identical to the double integral we would have had to calculate if we setup a double integral, over the triangle D in the xy plane, of the function defining the plane which makes up the top boundary of the tetrahedron.